ORIGIN ≡ 0

Testing Equal Variances in Two Populations: Variance Ratio F-test and Levine's Test

In order to decide whether to use two sample t-tests that assume equal variance between the populations, these tests are more-or-less useful depending on the underlying distribution of the data.

Variance Ratio F-Test for Equal Variances in Two Samples:

ZAR := READPRN("c:/DATA/Biostatistics/ZarEX8.7R.txt") **< Zar Example 8.7**

$i := 0..10$ $j := 11..20$

$X1_i := \left(ZAR^{\langle 1\rangle}\right)_i$ $X2_{j-11} := \left(ZAR^{\langle 1\rangle}\right)_j$

$n_1 := \text{length}(X1)$ $n_1 = 11$ $n_2 := \text{length}(X2)$ $n_2 = 10$

$X1_{bar} := \text{mean}(X1)$ $X1_{bar} = 36.4545$ $X2_{bar} := \text{mean}(X2)$ $X2_{bar} = 57.7$

$s_1 := \sqrt{\text{Var}(X1)}$ $s_1^2 = 21.872727$ $s_2 := \sqrt{\text{Var}(X2)}$ $s_2^2 = 12.9$

ZAR =

	0	1
0	1	41
1	2	35
2	3	33
3	4	36
4	5	40
5	6	46
6	7	31
7	8	37
8	9	34
9	10	30
10	11	38
11	12	52
12	13	57
13	14	62
14	15	55
15	16	64
16	17	57
17	18	56
18	19	55

$$X1 = \begin{pmatrix} 41 \\ 35 \\ 33 \\ 36 \\ 40 \\ 46 \\ 31 \\ 37 \\ 34 \\ 30 \\ 38 \end{pmatrix} \quad X2 = \begin{pmatrix} 52 \\ 57 \\ 62 \\ 55 \\ 64 \\ 57 \\ 56 \\ 55 \\ 60 \\ 59 \end{pmatrix}$$

Assumptions:

- **Observed values $X_{1,1}, X_{1,2}, X_{1,3}, \ldots X_{1,n1}$ are a random sample from $\sim N(\mu_1,\sigma_1^2)$**
- **Observed values $X_{2,1}, X_{2,2}, X_{2,3}, \ldots X_{2,n2}$ are a random sample from $\sim N(\mu_2,\sigma_2^2)$**
- **Samples from the two samples are *independent.***

^ Note: this test is severely compromised by non normal distributions. See Levene Test for a possibly better alternative.

Hypotheses:

H_0: $\sigma_1^2 = \sigma_2^2$ **< No difference in variance between populations X_1. & X_2.**

H_1: $\sigma_1^2 \neq \sigma_2^2$ < **Two Sided Case**

H_1: $\sigma_1^2 < \sigma_2^2$ < **One Sided Case Lower Tail)**

H_1: $\sigma_1^2 > \sigma_2^2$ < **One Sided Case (Upper Tail)**

Test Statistics:

Two Sided Case:

$$F := \frac{s_1^2}{s_2^2} \qquad F = 1.69556$$

< put larger variance in numerator here!

One Sided Case Lower Tail):

$$F_A := \frac{s_2^2}{s_1^2} \qquad F_A = 0.5898$$

One Sided Case (Upper Tail):

$$F_B := \frac{s_1^2}{s_2^2} \qquad F_B = 1.6956$$

Sampling Distribution:

If Assumptions hold and H_0 is true, then $F \sim F_{(n1-1)/(n2-1)}$

Critical Values of the Test:

$\alpha := 0.05$ **< Probability of Type I error must be explicitly set**

$C := qF\left(1 - \frac{\alpha}{2}, n_1 - 1, n_2 - 1\right)$ $C = 3.9639$ **< Two sided Critical Value**

$C_A := qF(\alpha, n_1 - 1, n_2 - 1)$ $C_A = 0.3311$ **< One sided lower Critical Value**

$C_B := qF(1 - \alpha, n_1 - 1, n_2 - 1)$ $C_B = 3.1373$ **< One sided upper Critical Value**

Decision Rules:

IF F > C, THEN REJECT H_0, OTHERWISE ACCEPT H_0 **< Two sided case**
IF $F_A < C_A$, THEN REJECT H_0, OTHERWISE ACCEPT H_0 **< One sided case lower tail**
IF $F_B > C_B$, THEN REJECT H_0, OTHERWISE ACCEPT H_0 **< One sided case upper tail**

Probability Values:

$P := 2 \cdot pF(F, n_1 - 1, n_2 - 1)$ $P = 1.5599$ **< if** $F \le 1$
< Two sided case
$P := 2 \cdot (1 - pF(F, n_1 - 1, n_2 - 1))$ $P = 0.4401$ **< if F > 1**

$P := pF(F, n_1 - 1, n_2 - 1)$ $P = 0.78$ **< One sided case lower tail**

$P := 1 - pF(F, n_1 - 1, n_2 - 1)$ $P = 0.22$ **< One sided case upper tail**

Confidence Intervals for Variance Ratio:

$$L_1 := \left(\frac{s_1^{\,2}}{s_2^{\,2}}\right) \cdot \left(\frac{1}{qF\left(1 - \frac{\alpha}{2}, n_1 - 1, n_2 - 1\right)}\right)$$ $L_1 = 0.4277543$

$\frac{s_1^{\,2}}{s_2^{\,2}} = 1.6956$

$$L_2 := \left(\frac{s_1^{\,2}}{s_2^{\,2}}\right) \cdot \left(qF\left(1 - \frac{\alpha}{2}, n_2 - 1, n_1 - 1\right)\right)$$ $L_2 = 6.4074588$ **< Two sided case**

$(L_1 \ \ L_2) = (0.4277543 \ \ 6.4074588)$

^ Note: CI for the inverse ratio: $s_2^{\,2}/s_1^{\,2}$, simply reverse indices 1 & 2 in the equations above.

$$L_3 := \left(\frac{s_1^{\,2}}{s_2^{\,2}}\right) \cdot \left(qF(1 - \alpha, n_2 - 1, n_1 - 1)\right)$$ $L_3 = 5.121241$ **< One sided case lower tail**

zero to L_3

$$L_4 := \left(\frac{s_1^{\,2}}{s_2^{\,2}}\right) \cdot \left(\frac{1}{qF(1 - \alpha, n_1 - 1, n_2 - 1)}\right)$$ $L_4 = 0.5404555$ **< One sided case upper tail**

L_4 to infinity

Prototype in R:

```
#READING DATA IN R FORMAT:
ZAR=read.table("c:/DATA/Biostatistics/ZarEX8.7R.txt")
ZAR
attach(ZAR)
X1=data[type=="typ1"]
X1
X2=data[type=="typ2"]
X2

#PERFORMING F TEST FOR EQUAL VARIANCES"
?var.test

#TWO-SIDED CASE:
var.test(X1,X2,alternative="two.sided",conf.level=0.95)
```

```
        F test to compare two variances

data:  X1 and X2
F = 1.6956, num df = 10, denom df = 9, p-value = 0.4401
alternative hypothesis: true ratio of variances is not equal to 1
95 percent confidence interval:
 0.4277543 6.4074588
sample estimates:
ratio of variances
          1.695560
```

```
#NOTE ALTERNATIVE COMMAND IN R
#FOR THE SAME TEST:
var.test(data~type,alternative="two.sided",conf.level=0.95)
```

```
        F test to compare two variances

data:  data by type
F = 1.6956, num df = 10, denom df = 9, p-value = 0.4401
alternative hypothesis: true ratio of variances is not equal to 1
95 percent confidence interval:
 0.4277543 6.4074588
sample estimates:
ratio of variances
          1.695560
```

```
#ONE-SIDED CASE LOWER TAIL:
var.test(data~type,alternative="less",conf.level=0.95)
```

```
        F test to compare two variances

data:  data by type
F = 1.6956, num df = 10, denom df = 9, p-value = 0.78
alternative hypothesis: true ratio of variances is less than 1
95 percent confidence interval:
 0.000000 5.121241
sample estimates:
ratio of variances
          1.695560
```

```
#ONE-SIDED CASE UPPER TAIL:
var.test(data~type,alternative="greater",conf.level=0.95)
```

```
        F test to compare two variances

data:  data by type
F = 1.6956, num df = 10, denom df = 9, p-value = 0.2200
alternative hypothesis: true ratio of variances is greater than 1
95 percent confidence interval:
 0.5404555       Inf
sample estimates:
ratio of variances
          1.695560
```

Levene Test for Equal Variances in Two Samples:

This test employs a *transformation* of the original data values to difference values around each mean.

Assumptions:

- **Observed values $X_{1,1}, X_{1,2}, X_{1,3}, \ldots X_{1,n1}$ are a random sample from $\sim N(\mu_1,\sigma_1^2)$**
- **Observed values $X_{2,1}, X_{2,2}, X_{2,3}, \ldots X_{2,n2}$ are a random sample from $\sim N(\mu_2,\sigma_2^2)$**
- **Samples from the two samples are *independent.***

^ Note: this test is somewhat less compromised by non-normal distributions.

$$X1 = \begin{pmatrix} 41 \\ 35 \\ 33 \\ 36 \\ 40 \\ 46 \\ 31 \\ 37 \\ 34 \\ 30 \\ 38 \end{pmatrix} \quad X2 = \begin{pmatrix} 52 \\ 57 \\ 62 \\ 55 \\ 64 \\ 57 \\ 56 \\ 55 \\ 60 \\ 59 \end{pmatrix}$$

Hypotheses:

$H_0: \sigma_1^2 = \sigma_2^2$ **< No difference in variance between populations $X_{1.}$ & $X_{2.}$**

$H_1: \sigma_1^2 \neq \sigma_2^2$ **< Two Sided Case**

$H_1: \sigma_1^2 < \sigma_2^2$ **< One Sided Case Lower Tail)**

$H_1: \sigma_1^2 > \sigma_2^2$ **< One Sided Case (Upper Tail)**

$X1_{bar} = 36.4545 \qquad X2_{bar} = 57.7$

Transformation to Absolute Difference from Mean:

$X1T := \overrightarrow{\left|X1 - X1_{bar}\right|}$

$X2T := \overrightarrow{\left|X2 - X2_{bar}\right|}$

< Each value of X1 and X2 is subtracted from the respective mean for X1 or X2, and then the absolute value is taken.

$X1T_{bar} := \text{mean}(X1T) \qquad X1T_{bar} = 3.5868$

$X2T_{bar} := \text{mean}(X2T) \qquad X2T_{bar} = 2.84$

$sT_1 := \sqrt{\text{Var}(X1T)} \qquad sT_1^2 = 7.721262 \qquad sT_2 := \sqrt{\text{Var}(X2T)} \qquad sT_2^2 = 3.938222$

$$X1T = \begin{pmatrix} 4.545 \\ 1.455 \\ 3.455 \\ 0.455 \\ 3.545 \\ 9.545 \\ 5.455 \\ 0.545 \\ 2.455 \\ 6.455 \\ 1.545 \end{pmatrix} \quad X2T = \begin{pmatrix} 5.7 \\ 0.7 \\ 4.3 \\ 2.7 \\ 6.3 \\ 0.7 \\ 1.7 \\ 2.7 \\ 2.3 \\ 1.3 \end{pmatrix}$$

$$sT_p := \sqrt{\frac{(n_1 - 1) \cdot sT_1^2 + (n_2 - 1) \cdot sT_2^2}{n_1 + n_2 - 2}} \qquad sT_p^2 = 5.9293$$

^ pooled variance of transformed variables

Test Statistics:

$$t := \frac{X1T_{bar} - X2T_{bar}}{\sqrt{\frac{sT_p^2}{n_1} + \frac{sT_p^2}{n_2}}} \qquad t = 0.7019 \qquad t^2 = 0.4927$$

Sampling Distribution:

If Assumptions hold and H_0 is true, then $t \sim t_{(n1+n2-2)}$

Critical Values of the Test:

$\alpha := 0.05$ < **Probability of Type I error must be explicitly set**

$C_1 := qt\left(\frac{\alpha}{2}, n_1 + n_2 - 2\right)$	$C_1 = -2.093$	**< Two sided lower Critical Value**
$C_2 := qt\left(1 - \frac{\alpha}{2}, n_1 + n_2 - 2\right)$	$C_2 = 2.093$	**< Two sided upper Critical Value**
$C := \lvert C_1 \rvert$	$C = 2.093$	**< Critical value used for two sided test (to simplify)**
$C_3 := qt(\alpha, n_1 + n_2 - 2)$	$C_3 = -1.7291$	**< One sided lower Critical Value**
$C_4 := qt(1 - \alpha, n_1 + n_2 - 2)$	$C_4 = 1.7291$	**< One sided upper Critical Value**

Decision Rules:

IF $|t| > C$ THEN REJECT H_0, OTHERWISE ACCEPT H_0 **< Two sided case**
IF $t < C_3$, THEN REJECT H_0, OTHERWISE ACCEPT H_0 **< One sided case lower tail**
IF $t > C_4$, THEN REJECT H_0, OTHERWISE ACCEPT H_0 **< One sided case upper tail**

Probability Values:

$t = 0.7019$

$P := 2 \cdot pt(t, n_1 + n_2 - 2)$	$P = 1.50875$	**< if** $t \le 0$	**< Two sided case**
$P := 2 \cdot (1 - pt(t, n_1 + n_2 - 2))$	$P = 0.49125$	**< if $t > 0$**	
$P := pt(t, n_1 + n_2 - 2)$	$P = 0.754375$		**< One sided case lower tail**
$P := 1 - pt(t, n_1 + n_2 - 2)$	$P = 0.245625$		**< One sided case upper tail**

Prototype in R:

```
#PERFORMING LEVENE TEST FOR EQUAL VARIANCES
#DOWNLOAD {car} FROM CRAN WEBSITE
library(car)
?leveneTest
leveneTest(data,type,center=mean)
leveneTest(data,type,center=median)
```

Thanks are due to Radovan Omorjan for correcting difficulties with a previous version of this worksheet!

```
> leveneTest(data,type,center=mean)
Levene's Test for Homogeneity of Variance (center = mean)
      Df F value Pr(>F)
group  1  0.4927 0.4913
      19
```

$t^2 = 0.4927$

```
> leveneTest(data,type,center=median)
Levene's Test for Homogeneity of Variance (center = median)
      Df F value Pr(>F)
group  1  0.5438 0.4699
      19
```

^ Results from R match the calculations above when center=mean option is chosen within leveneTest(). Note that leveneTest() reports an F statistic which is the square of the t statistic calculated above.

R documentation for levineTest() indicates that use of the median rather than mean for center results in a more robust test.