ORIGIN ≡ 0

Two Sample t-Test for Populations with Unequal Variances

This test is employed where two sets of measurements are derived from samples failing the F-test for equal variances or otherwise assumed to have unequal variances. The general situation is known as the Behrens-Fisher Problem, for which several statistical solutions have been offered.

ZAR := READPRN("c:/DATA/Biostatistics/ZarEX8.2R.txt") **< Zar Example 8.2**

$i := 0..9$ $j := 10..17$

$X1_i := \left(ZAR^{\langle 1 \rangle}\right)_i$ $X2_{j-10} := \left(ZAR^{\langle 1 \rangle}\right)_j$

$n_1 := \text{length}(X1)$ $n_1 = 10$ $n_2 := \text{length}(X2)$ $n_2 = 8$

$X1_{bar} := \text{mean}(X1)$ $X1_{bar} = 51.91$ $X2_{bar} := \text{mean}(X2)$ $X2_{bar} = 56.55$

$s_1 := \sqrt{\text{Var}(X1)}$ $s_1 = 3.370279$ $s_2 := \sqrt{\text{Var}(X2)}$ $s_2 = 3.144156$

ZAR =

	0	1
0	1	48.2
1	2	54.6
2	3	58.3
3	4	47.8
4	5	51.4
5	6	52
6	7	55.2
7	8	49.1
8	9	49.9
9	10	52.6
10	11	52.3
11	12	57.4
12	13	55.6
13	14	53.2
14	15	61.3
15	16	58
16	17	59.8
17	18	54.8

^ values for the two groups X1 & X2 read by indices *i* & *j* respectively
I counted the numbers from the original dataset.

Assumptions:

- **Observed values $X_{1,1}, X_{1,2}, X_{1,3}, \ldots X_{1,n1}$ are a random sample from $\sim N(\mu_1,\sigma_1^2)$**
- **Observed values $X_{2,1}, X_{2,2}, X_{2,3}, \ldots X_{2,n2}$ are a random sample from $\sim N(\mu_2,\sigma_2^2)$**
- **Variances σ_1^2 & σ_2^2 are *unequal* and *unknown.***
- **Samples $X_{1,n1}$ and $X_{2,n2}$ are *independent.***

^ Note: this test is reasonably robust for deviations from $\sim N(\mu,\sigma^2)$.

Hypotheses:

$H_0: \mu_1 = \mu_2$ **< No difference in mean between populations $X_{1.}$ & $X_{2.}$**

$H_1: \mu_1 \neq \mu_2$ **< Two Sided Case**

$H_1: \mu_1 < \mu_2$ **< One Sided Case Lower Tail)**

$H_1: \mu_1 > \mu 2$ **< One Sided Case (Upper Tail)**

$$X1 = \begin{pmatrix} 48.2 \\ 54.6 \\ 58.3 \\ 47.8 \\ 51.4 \\ 52 \\ 55.2 \\ 49.1 \\ 49.9 \\ 52.6 \end{pmatrix} \quad X2 = \begin{pmatrix} 52.3 \\ 57.4 \\ 55.6 \\ 53.2 \\ 61.3 \\ 58 \\ 59.8 \\ 54.8 \end{pmatrix}$$

Test Statistic:

$$t := \frac{X1_{bar} - X2_{bar}}{\sqrt{\dfrac{s_1^2}{n_1} + \dfrac{s_2^2}{n_2}}} \qquad t = -3.013$$

< t is the normalized mean X_1bar - X_2bar

Satterthwaite's Method Degrees of Freedom:

$$d_S := \frac{\left(\dfrac{s_1^2}{n_1} + \dfrac{s_2^2}{n_2}\right)^2}{\dfrac{\left(\dfrac{s_1^2}{n_1}\right)^2}{(n_1 - 1)} + \dfrac{\left(\dfrac{s_2^2}{n_2}\right)^2}{(n_2 - 1)}} \qquad d_S = 15.5587$$

Sampling Distribution:

If Assumptions hold and H_0 is true, then $t \sim t_{(dS)}$

Critical Values of the Test:

$\alpha := 0.05$ **< Probability of Type I error must be explicitly set**

$C_1 := qt\left(\frac{\alpha}{2}, d_S\right)$ $C_1 = -2.1248$ **< Two sided lower Critical Value**

$C_2 := qt\left(1 - \frac{\alpha}{2}, d_S\right)$ $C_2 = 2.1248$ **< Two sided upper Critical Value**

$C := \left|C_1\right|$ $C = 2.1248$ **< Critical value used for two sided test (to simplify)**

$C_3 := qt(\alpha, d_S)$ $C_3 = -1.7489$ **< One sided lower Critical Value**

$C_4 := qt(1 - \alpha, d_S)$ $C_4 = 1.7489$ **< One sided upper Critical Value**

Decision Rules:

IF |t| > C THEN REJECT H_0, OTHERWISE ACCEPT H_0 **< Two sided case**
IF t < C_3, THEN REJECT H_0, OTHERWISE ACCEPT H_0 **< One sided case lower tail**
IF t > C_4, THEN REJECT H_0, OTHERWISE ACCEPT H_0 **< One sided case upper tail**

Probability Values:

$t = -3.013$

$P := 2 \cdot pt(t, d_S)$ $P = 0.008458$ **< if** $t \le 0$ **< Two sided case**

$P := 2 \cdot (1 - pt(t, d_S))$ $P = 1.991542$ **< if t > 0**

$P := pt(t, d_S)$ $P = 0.004229$ **< One sided case lower tail**

$P := 1 - pt(t, d_S)$ $P = 0.995771$ **< One sided case upper tail**

Confidence Intervals for the mean:

$$CI := \left(X1_{bar} - X2_{bar} - C \cdot \sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}} \quad X1_{bar} - X2_{bar} + C \cdot \sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}} \right)$$

$X1_{bar} - X2_{bar} = -4.64$

$CI = (-7.912191 \quad -1.367809)$ **< Two sided case**

$$CIL := X1_{bar} - X2_{bar} - C_3 \cdot \sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}$$

minus infinity to $CIL = -1.946658$

^ One sided case lower tail

$$CIU := X1_{bar} - X2_{bar} - C_4 \cdot \sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}$$

$CIU = -7.333342$ **to infinity** **< One sided case upper tail**

Prototype in R:

```
#READING DATA IN R FORMAT:

ZAR=read.table("c:/DATA/Biostatistics/ZarEX8.2R.txt")
ZAR
attach(ZAR)
X1=data[group=="fer"]
X1
X2=data[group=="nofer"]
X2

#PERFORMING TWO-SAMPLE TWO SIDED t-TEST
#FOR UNEQUAL VARIANCES:

t.test(X1,X2,alternative="two.sided",var.equal=FALSE, conf.level=0.95)
```

```
        Welch Two Sample t-test

data:  X1 and X2
t = -3.013, df = 15.559, p-value = 0.008458
alternative hypothesis: true difference in means is not equal to 0
95 percent confidence interval:
 -7.912191 -1.367809
sample estimates:
mean of x mean of y
    51.91     56.55
```

^ Note: R results match. R reports that it uses a test called "Welch's procedure". See Lecture Worksheet 240.