ORIGIN ≡ 1

Welch's F-Test for H_0: All $\alpha_i = 0$ in One-Way ANOVA with Fixed Effects Model

This test is a variant of the more usual F-test for cases when variance differs importantly between classes. Thus, this test seeks to get around the usual assumption that ε_{ij} are a random sample ~ $N(0,\sigma^2)$.

Data Structure:

k groups with not necessarily the same numbers of observations and different means.

Let index i,j indicate the ith column (treatment class) and jth row (object).

One-Way ANOVA					
	Treatment Classes:				
Objects (Replicates)	#1	#2	#3	…	#k
1					
2					
3					
…					
n	n1	n2	n3		nk
means:	X1bar	X2bar	X3bar	…	Xkbar

ZAR := READPRN("c:/DATA/Biostatistics/ZarEX10.3R.txt")

Zar's Example 10.3

$j := 1..6$

$X1_j := \left(ZAR^{\langle 2\rangle}\right)_j$

$X2_j := \left(ZAR^{\langle 2\rangle}\right)_{j+6}$

$X3_j := \left(ZAR^{\langle 2\rangle}\right)_{j+12}$

$X := ZAR^{\langle 2\rangle}$

ZAR =

	1	2	3
1	1	27.9	1
2	2	27	1
3	3	26	1
4	4	26.5	1
5	5	27	1
6	6	27.5	1
7	7	24.2	2
8	8	24.7	2
9	9	25.6	2
10	10	26	2
11	11	27.4	2
12	12	26.1	2
13	13	29.1	3
14	14	27.7	3
15	15	29.9	3
16	16	30.7	3
17	17	28.8	3
18	18	31.1	3

$$X1 = \begin{pmatrix} 27.9 \\ 27 \\ 26 \\ 26.5 \\ 27 \\ 27.5 \end{pmatrix}$$

$X1_{bar} := mean(X1)$ $\quad X1_{bar} = 26.9833$

$n_1 := length(X1)$ $\quad n_1 = 6$

$s_1 := \sqrt{Var(X1)}$ $\quad s_1 = 0.6795$

$$X2 = \begin{pmatrix} 24.2 \\ 24.7 \\ 25.6 \\ 26 \\ 27.4 \\ 26.1 \end{pmatrix}$$

$X2_{bar} := mean(X2)$ $\quad X2_{bar} = 25.6667$

$n_2 := length(X2)$ $\quad n_2 = 6$

$s_2 := \sqrt{Var(X2)}$ $\quad s_2 = 1.1308$

$$X3 = \begin{pmatrix} 29.1 \\ 27.7 \\ 29.9 \\ 30.7 \\ 28.8 \\ 31.1 \end{pmatrix}$$

$X3_{bar} := mean(X3)$ $\quad X3_{bar} = 29.55$

$n_3 := length(X3)$ $\quad n_3 = 6$

$s_3 := \sqrt{Var(X3)}$ $\quad s_3 = 1.2677$

index variables:
j indexes individual cases within classes and ranges from 1 to n_j for each group.
i indexes classes from 1 to k,
k is the total number of classes.

$k := 3$

$i := 1..k$

$N := length\left(ZAR^{\langle 1\rangle}\right)$ $\quad N = 18$

$$X_{bar} := \begin{pmatrix} X1_{bar} \\ X2_{bar} \\ X3_{bar} \end{pmatrix} \quad X_{bar} = \begin{pmatrix} 26.9833 \\ 25.6667 \\ 29.55 \end{pmatrix} \quad n := \begin{pmatrix} n_1 \\ n_2 \\ n_2 \end{pmatrix} \quad n = \begin{pmatrix} 6 \\ 6 \\ 6 \end{pmatrix} \quad s := \begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix} \quad s = \begin{pmatrix} 0.6794606 \\ 1.1307814 \\ 1.267675 \end{pmatrix}$$

< summary statistics

Welch's F Test for Overall Comparison of Class Means:

Treatments Model:

$X_{i,j} = \mu + \alpha_i + \varepsilon_{i,j}$ < where:

μ is the grand mean of all objects.
α_i is the mean of i = $\mu + \alpha_i$ for each class i.
$\varepsilon_{i,j}$ is the error term specific to each object i,j

Restriction:

$\sum_i n_i \cdot \alpha_i := 0$ or $\sum_i \alpha_i := 0$ or $\alpha_k := 0$ < allows estimation of k parameters. Other restrictions are also possible:

Assumptions:

ε_{ij} are a random sample ~ $N(0,\sigma_i^2)$

^Errors come from different normal populations with mean zero but different variances.

One-Way ANOVA Table:

```
#ONE-WAY ANOVA TABLE
#WELCH'S F-TEST FOR ALL MEANS

ZAR=read.table("c:/DATA/Biostatistics/ZarEX10.3R.txt")
ZAR
attach(ZAR)

Y=potassium
X=factor(variety)

options(digits=12)
anova(lm(Y~X))
```

Calculation in R:

```
Analysis of Variance Table

Response: Y
          Df       Sum Sq      Mean Sq  F value      Pr(>F)
X          2 46.80333333 23.40166667 20.97341 4.5151e-05 ***
Residuals 15 16.73666667  1.11577778
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

One-Way ANOVA Table:

Values taken from R's ANOVA table:

Source:	SS	df		MS	
Between	$SS_B := 46.80333333$	$df_B := k - 1$	$df_B = 2$	$MS_B := \frac{SS_B}{df_B}$	$MS_B = 23.4017$
Within	$SS_W := 16.73666667$	$df_W := N - k$	$df_W = 15$	$MS_W := \frac{SS_W}{df_W}$	$MS_W = 1.1158$
TOTAL	$SS_T := SS_B + SS_W$				$k = 3$
	$SS_T = 63.54$				$N = 18$

Welch's Test:

Hypotheses:

H_0: $\alpha_i = 0$ *for all* i < All treatment class deviations from the grand mean are 0
H_1: *At least one* $\alpha_i \neq 0$ < **One sided test**

Test Statistic:

Previously calculated:

$k = 3$ $n = \begin{pmatrix} 6 \\ 6 \\ 6 \end{pmatrix}$

$X_{bar} = \begin{pmatrix} 26.9833 \\ 25.6667 \\ 29.55 \end{pmatrix}$ $s^2 = \begin{pmatrix} 0.4617 \\ 1.2787 \\ 1.607 \end{pmatrix}$

$i := 1 .. k$

$c_i := \dfrac{n_i}{(s_i)^2}$ $c = \begin{pmatrix} 12.9964 \\ 4.6924 \\ 3.7337 \end{pmatrix}$

$C := \sum c$ $C = 21.4224$

$X_{barW} := \dfrac{\sum_i c_i \cdot X_{bar_i}}{C}$ $X_{barW} = 27.1423$

$A := \sum_i \dfrac{\left(1 - \frac{c_i}{C}\right)^2}{n_i - 1}$ $A = 0.2893$

$F_W := \dfrac{\sum_i c_i \cdot \left(X_{bar_i} - X_{barW}\right)^2}{(k-1) \cdot \left[1 + \frac{2 \cdot A \cdot (k-2)}{k^2 - 1}\right]}$ $F_W = 15.0096$

< This number doesn't match Zar p. 204, but is identical with R's oneway.test() results below.

Distribution of the test Statistic F:

If H_0 is true then $F_W \sim F_{((k-1),\nu 2)}$ **where: k = number of classes**

$\nu_2 = (k^2-1)/3A$

$\nu_2 := \dfrac{k^2 - 1}{3 \cdot A}$ $\nu_2 = 9.2183$

Critical Value of the Test:

$\alpha := 0.05$ **< Probability of Type I error must be explicitly set**

$C := qF[1 - \alpha, (k-1), \nu_2]$ $C = 4.2195$

Decision Rule:

IF Fw > C, THEN REJECT H_0, OTHERWISE ACCEPT H_0

Probability Value:

$P := 1 - pF[F_W, (k-1), \nu_2]$ $P = 0.001261$

Prototype in R:

```
?oneway.test
oneway.test(Y~X,var.equal=FALSE)
```

```
One-way analysis of means (not assuming equal variances)

data:  Y and X
F = 15.0096, num df = 2.000, denom df = 9.218, p-value = 0.001261
```