ORIGIN ≡ 1

Hotelling's T^2 Test for a Paired Design

Just as with the univariate t-test, Hotelling's T^2 test for paired data is fundamentally identical to the corresponding one-population version of the test. All one does is use the difference D matrix between matched observations X and Y and treat this as a single population. As with the univariate case, paired tests have greater power than non-paired two-sample T^2 tests because variance between individuals, unrelated to treatments, is better controlled. The example comes from RA Johnson & DW Wichern (JW) *Applied Multivariate Statistical Analysis 4th Edition* 1998. It is useful in demonstrating that rejection of the null hypothesis is possible even when all confidence intervals include the vector of zero difference. In these cases, the multivariate T^2 confidence ellipsoid, identical with the T^2 test, should be preferred.

Effluent Data JW Table 6-1 p. 275)
Samples taken and split for comparison of technique between:

X = variables BOD & SS measured in a Commercial Lab.
Y = variables BOD & SS measured in a State Lab.

Read Data:

M := READPRN("c:\DATA\Multivariate\T6-1.DAT")

n := rows(M) n = 11

X := submatrix(M, 1, n, 1, 2)

Y := submatrix(M, 1, n, 3, 4)

p := cols(X) p = 2

M =

	1	2	3	4
1	6	27	25	15
2	6	23	28	13
3	18	64	36	22
4	8	44	35	29
5	11	30	15	31
6	34	75	44	64
7	28	26	42	30
8	71	124	54	64
9	43	54	34	56
10	33	30	29	20
11	20	14	39	21

X =

	1	2
1	6	27
2	6	23
3	18	64
4	8	44
5	11	30
6	34	75
7	28	26
8	71	124
9	43	54
10	33	30
11	20	14

Y =

	1	2
1	25	15
2	28	13
3	36	22
4	35	29
5	15	31
6	44	64
7	42	30
8	54	64
9	34	56
10	29	20
11	39	21

Calculate Difference Matrix:

D := X − Y

D =

	1	2
1	-19	12
2	-22	10
3	-18	42
4	-27	15
5	-4	-1
6	-10	11
7	-14	-4
8	17	60
9	9	-2
10	4	10
11	-19	-7

Summary Statistics:

i := 1 .. n j := 1 .. p

I := identity(n) $l_{n_i} := 1$

Mean Vectors:

$X_{bar} := \frac{1}{n} \cdot X^T \cdot l_n$ $X_{bar} = \begin{pmatrix} 25.272727 \\ 46.454545 \end{pmatrix}$

$Y_{bar} := \frac{1}{n} \cdot Y^T \cdot l_n$ $Y_{bar} = \begin{pmatrix} 34.636364 \\ 33.181818 \end{pmatrix}$

$D_{bar} := \frac{1}{n} \cdot D^T \cdot l_n$ $D_{bar} = \begin{pmatrix} -9.363636 \\ 13.272727 \end{pmatrix}$

Covariance Matrices:

$S_X := \frac{1}{n-1} \cdot X^T \cdot \left(I - \frac{1}{n} \cdot l_n \cdot l_n^T \right) \cdot X$ $S_X = \begin{pmatrix} 387.41818 & 489.36364 \\ 489.36364 & 1014.07273 \end{pmatrix}$

$S_Y := \frac{1}{n-1} \cdot Y^T \cdot \left(I - \frac{1}{n} \cdot l_n \cdot l_n^T \right) \cdot Y$ $S_Y = \begin{pmatrix} 109.25455 & 120.37273 \\ 120.37273 & 363.76364 \end{pmatrix}$

$S_D := \frac{1}{n-1} \cdot D^T \cdot \left(I - \frac{1}{n} \cdot l_n \cdot l_n^T \right) \cdot D$ $S_D = \begin{pmatrix} 199.25455 & 88.30909 \\ 88.30909 & 418.61818 \end{pmatrix}$

Assumptions:

- Observed values $D_1, D_2, D_3, \ldots D_n$ are a random sample from $\sim N_p(\mu,\Sigma_D)$.
- Covaraince Matrix Σ of the population is *unknown* and estimated by S_D.

Hypotheses:

Specify a test vector μ_0:

^ Note: this test is reasonably robust for deviations from $\sim N_p(\mu,\Sigma_D)$ with sufficient sample size

$$\delta_o := \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

H_0: $\delta = \delta_0$ < δ_0 (0,0) or any value of difference between X & Yis a specified value for δ

H_1: $\delta \neq \delta_0$ < Note: Unlike univariate t-tests, All Hotelling T^2 tests are two-sided since the T^2 distribution, or equivalent F distribution, is asymmetric.

Hotelling's Test Statistic:

$$T_{sq} := n\cdot\left(D_{bar} - \delta_o\right)^T\cdot\left(S_D\right)^{-1}\cdot\left(D_{bar} - \delta_o\right) \qquad T_{sq} = (13.639312)$$

^ T^2 is the normalized distance between vectors D_{bar} and δ_0

Likelihood Ratio Test Statistic:

Maximum likelihood estimates of μ & Σ (JW Result 4.11 p. 171) and Σ_0:

$$\mu_{hat} := X_{bar}$$

$$\Sigma_{hat} := \frac{(n-1)}{n}\cdot S_D \qquad \left|\Sigma_{hat}\right| = 6.249 \times 10^4$$

$$\Sigma_{0.hat} := \sum_i \left[\left(X^T\right)^{\langle i\rangle} - \delta_o\right]\cdot\left[\left(X^T\right)^{\langle i\rangle} - \delta_o\right]^T \qquad \left|\Sigma_{0.hat}\right| = 5.216 \times 10^7$$

Likelihood Ratio & Wilks' lambda (Λ) (JW Eq. 5-13 p. 217):

$$\Lambda := \left(\frac{\left|n\cdot\Sigma_{hat}\right|}{\left|\Sigma_{0.hat}\right|}\right)^{\frac{n}{2}} \qquad \Lambda = 0.00002438 \qquad \Lambda^{\frac{2}{n}} = 0.144974$$

< Λ = Wilks' lambda - value is 1.0 when $X_{bar} = \mu_0$ but decreases as μ_0 increases in distance from the sample mean.

Converting to Hotellings T^2:

$$\left[1 + \frac{T_{sq}}{(n-1)}\right]^{-1} = (0.423024)$$

< Equivalent value in terms of Hotellings T^2 (jw Result 5.1 p. 218)

$$\left[\Lambda^{-\left(\frac{2}{n}\right)} - 1\right]\cdot(n-1) = 58.978 \qquad T_{sq} = (13.639)$$

< Solving for T^2 (same as T^2 above)

Sampling Distribution:

If Assumptions hold and H_0 is true, then $T_{sq} \sim T^2_{(n-1)} = [(n-1)p/(n-p)]\, F_{(p,n-p)}$

Critical Value of the Test:

$\alpha := 0.05$ **< Probability of Type I error must be explicitly set**

$C := \frac{(n-1)\cdot p}{(n-p)}\cdot qF(1-\alpha, p, n-p)$ $C = 9.459$ **< JW eq. 5-6 p. 212**

NOTE: qF(1-α) is used in function qF.

Decision Rule:

Reject H_0 if $T_{sq} > C$

$T_{sq} = (13.639)$ $C = 9.4589$

$Decision := if(T_{sq_1} > C, 1, 0)$ $Decision = 1$ **< 0 = Do not reject H_0**

1 = Reject H_0

^ Therefore DO NOT Reject H_0

Probability Value:

$P := 1 - pF\left[T_{sq}\cdot\frac{(n-p)}{(n-1)\cdot p}, p, n-p\right]$ $P = (0.02082779)$

The Multivariate Confidence Ellipsoid:

$\lambda := reverse(sort(eigenvals(S_D)))$

$\varepsilon^{\langle j \rangle} := eigenvec(S_D, \lambda_j)$

$$\lambda = \begin{pmatrix} 449.75041 \\ 168.12231 \end{pmatrix} \qquad \varepsilon = \begin{pmatrix} 0.33248 & -0.94311 \\ 0.94311 & 0.33248 \end{pmatrix}$$

< Coordinates of each column vector of ε gives the directions of confidence elipsoid

$CT := \sqrt{\frac{p\cdot(n-1)}{n\cdot(n-p)}\cdot qF(1-\alpha, p, n-p)}$ $CT = 0.9273068$ **< CT gives the boundary for the confidence ellipsoid for μ - see JW eq. 5-18 p. 221**

$i := 1 .. p$

$L_i := CT\cdot\sqrt{\lambda_i}$

Multivariate confidence ellipsoid (JW Eq. 5-18 p. 221):

$$D_{bar} = \begin{pmatrix} -9.364 \\ 13.273 \end{pmatrix}$$ **< Center of ellipsoid** $$L = \begin{pmatrix} 19.66569 \\ 12.02364 \end{pmatrix}$$

< L are half-lengths of the axes of the confidence ellipsoid for m in the directions of ε

Simultaneous T^2 Confidence Intervals:

$CI_{lower_i} := D_{bar_i} - \sqrt{C}\cdot\sqrt{\frac{S_{D_{i,i}}}{n}}$ $CI_{upper_i} := D_{bar_i} + \sqrt{C}\cdot\sqrt{\frac{S_{D_{i,i}}}{n}}$ **< jw eq. 5-24 p. 225 (slightly modified)**

$CI := augment(CI_{lower}, CI_{upper})$

Simultaneous Confidence Intervals:

$$D_{bar} = \begin{pmatrix} -9.36364 \\ 13.27273 \end{pmatrix}$$ **< Mean values** $$CI = \begin{pmatrix} -22.45327 & 3.726 \\ -5.70012 & 32.24557 \end{pmatrix}$$ **< T^2 confidence intervals**

Univariate t Confidence Intervals:

$ct := qt\left(1 - \frac{\alpha}{2}, n - 1\right)$ $ct = 2.228139$ < Critical value ct based on t distribution without correction

$ci_{lower_i} := D_{bar_i} - ct \cdot \sqrt{\frac{S_{D_{i,i}}}{n}}$ $ci_{upper_i} := D_{bar_i} + ct \cdot \sqrt{\frac{S_{D_{i,i}}}{n}}$ < jw eq. 5-29 p. 232

$ci := augment(ci_{lower}, ci_{upper})$

Univariate t Intervals:

$D_{bar} = \begin{pmatrix} -9.36364 \\ 13.27273 \end{pmatrix}$ < Mean values $ci = \begin{pmatrix} -18.84673 & 0.11946 \\ -0.4726 & 27.01805 \end{pmatrix}$ < Univariate t confidence intervals

Bonferroni Simultaneous Confidence Intervals:

$cb := qt\left(1 - \frac{\alpha}{2 \cdot p}, n - 1\right)$ $cb = 2.633767$ < Critical value cb based on t distribution with Bonferroni correction factor p.

$ci_{lower_i} := D_{bar_i} - cb \cdot \sqrt{\frac{S_{D_{i,i}}}{n}}$ $ci_{upper_i} := D_{bar_i} + cb \cdot \sqrt{\frac{S_{D_{i,i}}}{n}}$ < jw eq. 5-29 p. 232

$ci := augment(ci_{lower}, ci_{upper})$

Bonferroni Intervals:

$D_{bar} = \begin{pmatrix} -9.36364 \\ 13.27273 \end{pmatrix}$ < Mean values $ci = \begin{pmatrix} -20.573107 & 1.845835 \\ -2.974903 & 29.520358 \end{pmatrix}$ < Bonferroni confidence intervals

Prototype in R:

```
#HOTELLING'S T2 TEST FOR PAIRED DESIGN:

#LOAD DATA & HYPOTHESIS:
M=read.table("c:/DATA/Multivariate/JW6-1-longform.txt")
M  #DATA TABLE

#SEPARATING GROUPS:
attach(M)
X=cbind(BOD=BOD[Lab==1],SS=SS[Lab==1])
X
Y=cbind(BOD=BOD[Lab==2],SS=SS[Lab==2])
Y
detach(M)
```

```
> M  #DATA TABLE
   BOD  SS Lab
1    6  27   1
2    6  23   1
3   18  64   1
4    8  44   1
5   11  30   1
6   34  75   1
7   28  26   1
8   71 124   1
9   43  54   1
10  33  30   1
11  20  14   1
12  25  15   2
13  28  13   2
14  36  22   2
15  35  29   2
16  15  31   2
17  44  64   2
18  42  30   2
19  54  64   2
20  34  56   2
21  29  20   2
22  39  21   2
```

Note the data structure in R is the so-called "long form". This is by far the most common way to handle data comprising multiple groups. The "dummy variable" (or factor) Lab is used to indicate group membership, with each variable comprising only a single long column.

#FUNCTION FOR HOTELLING'S T2 TEST FOR PAIRED DESIGN:
#X = dataset 1
#Y = dataset 2
#mu0 = hypothesis vector
#alpha = alpha of the test

PD.Hotelling.T2 <- function(X,Y,mu0,alpha)
... body of function in R script ...
PD.Hotelling.T2(X,Y,mu0=c(0,0),alpha=0.05)
RES=PD.Hotelling.T2(X,Y,mu0=c(0,0),alpha=0.05)
RES

> X

	BOD	SS
[1,]	6	27
[2,]	6	23
[3,]	18	64
[4,]	8	44
[5,]	11	30
[6,]	34	75
[7,]	28	26
[8,]	71	124
[9,]	43	54
[10,]	33	30
[11,]	20	14

> Y

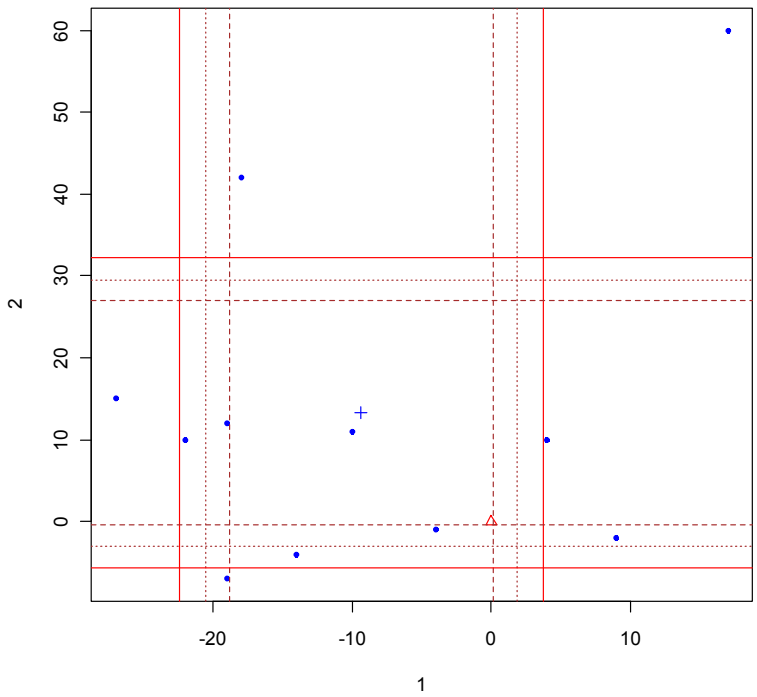
	BOD	SS
[1,]	25	15
[2,]	28	13
[3,]	36	22
[4,]	35	29
[5,]	15	31
[6,]	44	64
[7,]	42	30
[8,]	54	64
[9,]	34	56
[10,]	29	20
[11,]	39	21

> RES=PD.Hotelling.T2(X,Y,mu0=c(0,0),alpha=0.05)

```
Paired Design Hotelling's T2

Hypothesis Vector:       ( 0 0 )
Difference Vector        ( -9.363636 13.27273 )
Discriminant vector  :   ( -0.0673414 0.04591197 )
T2 Ellipsoid half lengths: ( 19.66569 12.02364 )
Hotelling's T2 Statistic:  13.63931
Equivalent F Statistic :   6.13769
F degrees of freedom:      ( 2 9 )
Wilks's Lambda:            0.008810659
alpha:                     0.05
Critical Value:            9.458877
Probability:               0.02082779

Confidence Intervals: T2 - Bonferroni - t - Mean - t - Bonferroni - T2

     T2.lower    B.lower     t.lower      Mean   t.upper   B.upper T2.upper
BOD -22.453272 -20.573107 -18.8467298 -9.363636  0.119457  1.845835  3.72600
SS   -5.700119  -2.974903  -0.4725958 13.272727 27.018050 29.520358 32.24557
```

#FUNCTION PLOT CI:
#R = return from function OS.Hotelling.T2 as a list
#X = variable to plot along the X axis
#Y = variable to plot along the Y axis
plot.CI <- function(R,X,Y)
... body of function in R script ...

plot.CI(RES,1,2)



Simultaneous T^2 intervals (solid red) are the widest, univariate t intervals (dashed) are the narrowest, with Bonferroni intervals (dotted) intermediate. Blue cross is sample mean vector. red triangle is hypothesis vector: