ORIGIN ≡ 1

MATRIX ALBEGRA: DEFINITIONS AND BASIC FUNCTIONS

Basic functions of matrix algebra with R prototypes are presented here. For more complete information, see RA Johnson & DW Wichern *Applied Multivariate Statistical Analysis 4th Edition* 1998 or AC. Rencher *Methods of Multivariate Analysis* 1995.

Data Types:

Scalar:

$a := 5 \quad b := 8 \quad c := 3.14$

Vector:

$$x_1 := \begin{pmatrix} 3 \\ -1 \\ 4 \end{pmatrix} \qquad x_2 := \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}$$

Matrix:

$$A := \begin{pmatrix} 0 & 3 & 1 \\ 1 & -1 & 1 \end{pmatrix} \qquad B := \begin{pmatrix} 1 & -2 & -3 \\ 2 & 5 & 1 \end{pmatrix}$$

Prototype in R:

```
#MATRIX ALGEBRA: DEFINITIONS AND BASIC FUNCTIONS
#SCALAR:
a=5;b=8;c=3.14
> a
[1] 5

#VECTOR:
x1=c(3,-1,4)
x2=c(1,2,-2)
> x1
[1]  3 -1  4

#MATRIX:
A=matrix(c(0,3,1,1,-1,1),nrow=2,ncol=3,byrow=T)

B=matrix(c(1,2,-2,5,-3,1),nrow=2,ncol=3,byrow=F,
 dimnames=list(c("row1","row2"),c("C.1","C.2","C.3")))

> A
     [,1] [,2] [,3]
[1,]    0    3    1
[2,]    1   -1    1

> B
     C.1 C.2 C.3
row1   1  -2  -3
row2   2   5   1
```

Vector:

Vector Addition & Scalar Multiplication:

$$x_1 + x_2 = \begin{pmatrix} 4 \\ 1 \\ 2 \end{pmatrix} \qquad c \cdot x_2 = \begin{pmatrix} 3.14 \\ 6.28 \\ -6.28 \end{pmatrix}$$

```
> #VECTOR ADDITION AND SCALAR MULTIPLICATION:
> x1+x2
[1] 4 1 2

> c*x2
[1]  3.14  6.28 -6.28
```

Vector Transpose & Multiplication:

$x_1^T = (3 \;\; -1 \;\; 4)$ < **tranpose**

$x_1 \cdot x_2 = -7$ < **dot product converted to a scalar**

$x_1^T \cdot x_2 = (-7)$ < **dot product**

$\overrightarrow{(x_1 \cdot x_2)} = \begin{pmatrix} 3 \\ -2 \\ -8 \end{pmatrix}$ < **product by row**

$x_1 \cdot x_2^T = \begin{pmatrix} 3 & 6 & -6 \\ -1 & -2 & 2 \\ 4 & 8 & -8 \end{pmatrix}$ < **cross product**

```
> #VECTOR YRANSPOSE & MULTIPLICATION:
> t(x1)       #transpose
     [,1] [,2] [,3]
[1,]    3   -1    4

> x1*x2       # product by row
[1]  3 -2 -8

> t(x1)%*%x2    # dot product
     [,1]
[1,]   -7

> x1%*%t(x2)    #cross product
> outer(x1,x2)  #Equivalent to line above
     [,1] [,2] [,3]
[1,]    3    6   -6
[2,]   -1   -2    2
[3,]    4    8   -8
```

In general, dot products are not commutative. Note how both Mathcad treats the distinction between scalar and vector dot products. In R, similar notation results in row by row multiplication. Usually, it is best to specify products using the transpose function, although notation for row versus column vector in R is somewhat cryptic.

Vector Length:

$\text{length}(x_1) = 3$ < **number of elements in a vector**

$L_{x1} := \left|\sqrt{x_1^T \cdot x_1}\right|$ $L_{x1} = 5.09902$

$|x_1| = 5.099$ ^ **Euclidean vector length**

^ **Note use of | | in MathCad also converts to scalars**

```
> #VECTOR LENGTH:

> length(x1)      #number of elements in a vector
[1] 3

> sqrt(t(x1)%*%x1)  #Euclidean length
         [,1]
[1,] 5.09902
```

Angle between Vectors:

$\cos_\theta := \left|\frac{x_1^T \cdot x_2}{|x_1| \cdot |x_2|}\right|$ $\cos_\theta = 0.4576043$

$\theta_r := \text{acos}(\cos_\theta)$ $\theta_r = 1.095497\,\text{rad}$

$\theta := \left(\frac{180}{\pi}\right) \cdot \theta_r$ $\theta = 62.767$

$\pi = 3.141593$

^ **Note use of built-in π functions** >

```
> #ANGLE BETWEEN VECTORS:
> cos.theta=(t(x1)%*%x2)/((sqrt(t(x1)%*%x1))*(sqrt(t(x2)%*%x2)))
> cos.theta
           [,1]
[1,] -0.4576043

> theta.rad=acos(cos.theta)
> theta.rad
         [,1]
[1,] 2.046095

> theta=(180/pi)*theta.rad
> theta
         [,1]
[1,] 117.2326
> pi
[1] 3.141593
```

When vectors are: PARALLEL, θ = 0 or 180 deg, & $\cos_\theta$ = 1 or -1.

PERPENDICULAR, θ = 90 or 270 deg, & $\cos_\theta$ = 0; also $x^T y = 0$

Projection of One Vector onto Another:

Projection of x_1 onto x_2:

$\text{proj}_{12} := \frac{\left|x_1^T \cdot x_2\right|}{\left|x_2^T \cdot x_2\right|} \cdot x_2$ $\text{proj}_{12} = \begin{pmatrix} 0.7777778 \\ 1.5555556 \\ -1.5555556 \end{pmatrix}$

Projection of x_2 onto x_1:

$\text{proj}_{21} := \frac{\left|x_2^T \cdot x_1\right|}{\left|x_1^T \cdot x_1\right|} \cdot x_1$ $\text{proj}_{21} = \begin{pmatrix} 0.8076923 \\ -0.2692308 \\ 1.0769231 \end{pmatrix}$

Note: this is a way to convert from one coordinate system to another.

```
> #PROJECTION OF VECTORS:

> proj12=((t(x1)%*%x2)/(t(x2)%*%x2))*x2
> proj12
[1] -0.7777778 -1.5555556  1.5555556

> proj21=((t(x2)%*%x1)/(t(x1)%*%x1))*x1
> proj21
[1] -0.8076923  0.2692308 -1.0769231

> #HANDY PROJECTION FUNCTION:
> #Projects x onto y
> proj <- function(x,y){
+ res=((t(x)%*%y)/(t(y)%*%y))*y
+ return(res)
+ }

> proj(x1,x2)
[1] -0.7777778 -1.5555556  1.5555556
> proj(x2,x1)
[1] -0.8076923  0.2692308 -1.0769231
```

Matrix:

Summary Information:

rows(A) = 2

cols(B) = 3

rank(A) = 2

^ **Matrix rank gives the number of linearly independent columns (or rows) of a matrix.**

```
> #MATRIX SUMMARY INFORMATION:
> nrow(A)
[1] 2
> ncol(B)
[1] 3
> qr(A)$rank
[1] 2
```

Matrix Transpose:

$$A = \begin{pmatrix} 0 & 3 & 1 \\ 1 & -1 & 1 \end{pmatrix} \qquad A^T = \begin{pmatrix} 0 & 1 \\ 3 & -1 \\ 1 & 1 \end{pmatrix}$$

< Changes row versus column orientation of the matrix.

Matrix Multiplication:

$A \cdot B = ▮ \qquad A^T \cdot B^T = ▮$

^ **Only conformable matrices & vectors can be multiplied**

$$A \cdot B^T = \begin{pmatrix} -9 & 16 \\ 0 & -2 \end{pmatrix}$$

$$A^T \cdot B = \begin{pmatrix} 2 & 5 & 1 \\ 1 & -11 & -10 \\ 3 & 3 & -2 \end{pmatrix}$$

$$B^T \cdot A = \begin{pmatrix} 2 & 1 & 3 \\ 5 & -11 & 3 \\ 1 & -10 & -2 \end{pmatrix}$$

$x_1^T \cdot A = ▮$

$$A \cdot x_1 = \begin{pmatrix} 1 \\ 8 \end{pmatrix}$$

$$x_2^T \cdot B^T = \begin{pmatrix} 3 & 10 \end{pmatrix}$$

Note: Multiplication is not commutative and here in Mathcad, you MUST use the Transpose function to multiply properly!

```
> #MATRIX MULTIPLICATION:
> A%*%B
Error in A %*% B : non-conformable
arguments
> t(A)%*%t(B)
Error in t(A) %*% t(B) : non-conformable
arguments
> A%*%t(B)
     row1 row2
[1,]   -9   16
[2,]    0   -2
> t(A)%*%B
     C.1 C.2 C.3
[1,]   2   5   1
[2,]   1 -11 -10
[3,]   3   3  -2
> t(B)%*%A
    [,1] [,2] [,3]
C.1    2    1    3
C.2    5  -11    3
C.3    1  -10   -2
> t(x1)%*%A
Error in t(x1) %*% A : non-conformable
arguments
> A%*%x1
     [,1]
[1,]    1
[2,]    8
> t(x2)%*%t(B)
     row1 row2
[1,]    3   10
```

Special kinds of Matrices:

$$S := \begin{pmatrix} 4 & 7 & 6 \\ 5 & 1 & 1 \\ 3 & 2 & 1 \end{pmatrix}$$

S = SQUARE MATRIX: a matrix, where rows(S) = cols(S)

$$M := \begin{pmatrix} 1 & -5 & 7 \\ -5 & 2 & 4 \\ 7 & 4 & 3 \end{pmatrix} \qquad M^T = \begin{pmatrix} 1 & -5 & 7 \\ -5 & 2 & 4 \\ 7 & 4 & 3 \end{pmatrix}$$

M = SYMMETRIC MATRIX: a kind of Square matrix, where $M = M^T$

$$D := \begin{pmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & -9 \end{pmatrix} \qquad D^T = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & -9 \end{pmatrix}$$

D = DIAGONAL MATRIX: a kind of Symmetric matrix, where $D_{ij} = 0$ for indices i & j not equal.

$$I := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad I^T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

I = IDENTITY MATRIX: a kind of Diagonal matrix where $I_{ii} = 1$.

```
> #IDENTITY MATRIX:
> I4=diag(4)
> I4
     [,1] [,2] [,3] [,4]
[1,]    1    0    0    0
[2,]    0    1    0    0
[3,]    0    0    1    0
[4,]    0    0    0    1
```

$$I_4 := \text{identity}(4) \qquad I_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$Q := \begin{pmatrix} \frac{-1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{-1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{-1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{-1}{2} \end{pmatrix}$$

Q = ORTHOGONAL MATRIX: An Orthogonal matrix has row (and column) vectors of unit length and perpendicular to each other, also meaning that their dot product is zero.

$$i := 1..\text{rows}(Q) \qquad j := 1..\text{rows}(Q) \qquad \left| Q^{\langle j \rangle} \right| = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

< each column is shown here to have unit length

Also, the Orthogonal matrix *inverse* is equal to its *transpose* (defined below).

$$Q^{-1} = \begin{pmatrix} -0.5 & 0.5 & 0.5 & 0.5 \\ 0.5 & -0.5 & 0.5 & 0.5 \\ 0.5 & 0.5 & -0.5 & 0.5 \\ 0.5 & 0.5 & 0.5 & -0.5 \end{pmatrix} \qquad Q^T = \begin{pmatrix} -0.5 & 0.5 & 0.5 & 0.5 \\ 0.5 & -0.5 & 0.5 & 0.5 \\ 0.5 & 0.5 & -0.5 & 0.5 \\ 0.5 & 0.5 & 0.5 & -0.5 \end{pmatrix}$$

Other Examples:

$$I_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad I_4^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad I_4^T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$Q_2 := \begin{pmatrix} 0.999998 & 0.001815 & -0.000364 \\ -0.000789 & 0.595486 & 0.803365 \\ -0.001675 & 0.803364 & -0.595486 \end{pmatrix} \qquad Q_2^{-1} = \begin{pmatrix} 1 & -0.0008 & -0.0017 \\ 0.0018 & 0.5955 & 0.8034 \\ -0.0004 & 0.8034 & -0.5955 \end{pmatrix} \qquad Q_2^T = \begin{pmatrix} 1 & -0.0008 & -0.0017 \\ 0.0018 & 0.5955 & 0.8034 \\ -0.0004 & 0.8034 & -0.5955 \end{pmatrix}$$

Matrix Trace:

$$S = \begin{pmatrix} 4 & 7 & 6 \\ 5 & 1 & 1 \\ 3 & 2 & 1 \end{pmatrix} \qquad \mathrm{tr}(S) = 6$$

The trace of a matrix is the sum of elements along the main diagonal.

```
> #MATRIX TRACE:
> is.square <- function(M) {
+ return (is.matrix(M) && (nrow(M) == ncol(M)));
+ }
> trace <- function(M) {
+ return (ifelse(is.square(M),sum(diag(M)),NA));
+ }
> trace(S)
[1]  6
```

Matrix Inverse:

$B^{-1} =$ ■ $\qquad (B^T)^{-1} =$ ■ $\qquad \begin{pmatrix} \mathrm{rows}(B) \\ \mathrm{cols}(B) \\ \mathrm{rank}(B) \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix}$

A matrix inverse only exists for Square matrices in which the number of rows or columns is the same as matrix rank. Such matrices are called NONSINGULAR. A matrix that fails this condition is termed SINGULAR

$$S = \begin{pmatrix} 4 & 7 & 6 \\ 5 & 1 & 1 \\ 3 & 2 & 1 \end{pmatrix} \quad S^{-1} = \begin{pmatrix} -0.0416667 & 0.2083333 & 0.0416667 \\ -0.0833333 & -0.5833333 & 1.0833333 \\ 0.2916667 & 0.5416667 & -1.2916667 \end{pmatrix} \quad \begin{pmatrix} \mathrm{rows}(S) \\ \mathrm{cols}(S) \\ \mathrm{rank}(S) \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix}$$

$$S \cdot S^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The inverse of a square matrix is the matrix such that when multiplied with the original matrix yields the Identity matrix.

```
> #MATRIX INVERSE:
> S=matrix(c(4,7,6,5,1,1,3,2,1),nrow=3,ncol=3,byrow=T)
> Sinv=solve(S)
> Sinv
            [,1]        [,2]         [,3]
[1,] -0.04166667  0.2083333  0.04166667
[2,] -0.08333333 -0.5833333  1.08333333
[3,]  0.29166667  0.5416667 -1.29166667
```

$$D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & -9 \end{pmatrix} \quad D^{-1} = \begin{pmatrix} 0.5 & 0 & 0 \\ 0 & 0.25 & 0 \\ 0 & 0 & -0.111 \end{pmatrix} \quad D \cdot D^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Whereas some matrix inverses (such as for diagonal matrices shown here are easy to determine most require extensive calculations that are best left to a computer algorithm. However, due to rounding errors in calculating matrix inverses, it is a wise precaution to calculate the Identity products as a check for bad behavior.

Quadratic Forms:

$$x_1^T \cdot M \cdot x_1 = (225)$$

Quadratic forms x^TMx for some vector x and matrix M are scalars that result from adding squared or cross-product terms of x.

If $x^TMx > 0$ then the quadratic form & Matrix M are called POSITIVE DEFINITE.
A Matrix M is Positive Definite if and only if M's Eigenvalues are all positive.

If $0 < D_{sq} = x^TMx$, then the quadratic form x^TMx may be viewed as squared distance of the data poin in vector x from the center of the coordinate system after performing a Linear Transformation.

If Matrix M = I the Identity Matrix, then D_{sq} involves EUCLIDEAN DISTANCES (squared).

```
> #QUADRATIC FORM:
> M=matrix(c(1,-5,7,-5,2,4,7,4,3),nrow=3,ncol=3,byrow=T)
> t(x1)%*%M%*%x1
     [,1]
[1,]  225
```