

ORIGIN ≡ 1

## Eigenvectors and Eigenvalues

This worksheet describes the algebra used to calculate "principal" or "characteristic" directions called **Eigenvectors** and the "principal" or "characteristic" values called **Eigenvalues** associated with these directions. The cases considered here are bivariate so they can be easily calculated and visualized as graphs. However, calculations are typically extended into multivariate hyperspace, the common condition in multivariate statistics. In all cases, there are as many eigenvector/eigenvalue pairs as there are dimensions in the multivariate data so long as the data are of "full rank" (i.e., not redundant because some variables are linear combinations of the others). For extended discussion, see RA Johnson & DW Wichern *Applied Multivariate Statistical Analysis 4th Edition 1998*, or AC. Rencher *Methods of Multivariate Analysis 1995*.

Consider a linear transformation from  $r$  to  $R$ :

$$\begin{aligned} x_R &= a \cdot x_r + b \cdot y_r \\ y_R &= c \cdot x_r + d \cdot y_r \end{aligned} \quad \text{or} \quad \begin{pmatrix} x_R \\ y_R \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} x_r \\ y_r \end{pmatrix} \quad < \quad R = M \cdot r$$

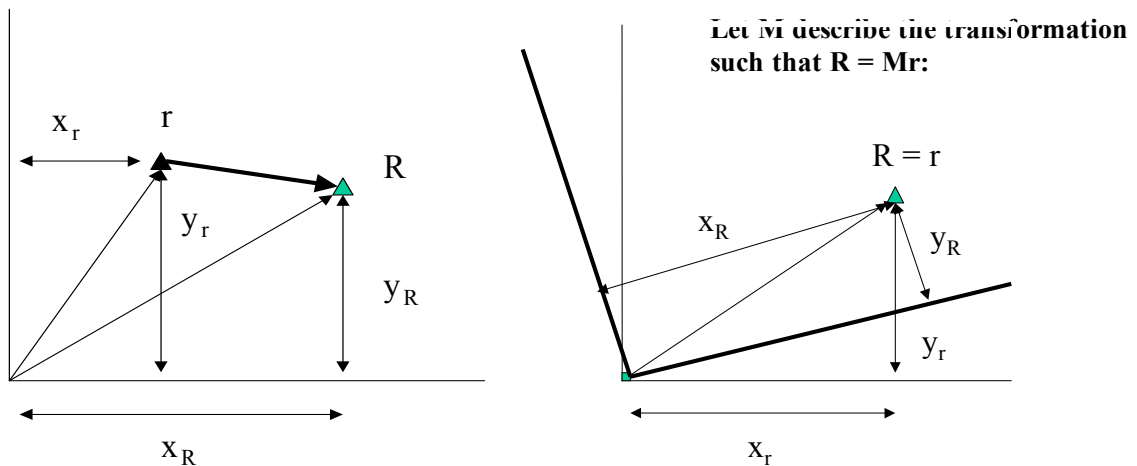
For example, given a linear transformation specified by some matrix  $M$  and data in vector  $X$ :

$$M := \begin{pmatrix} 5 & -2 \\ -3 & 2 \end{pmatrix} \quad r := \begin{pmatrix} 2 \\ 2 \end{pmatrix} \quad \begin{aligned} a &:= 5 & b &:= -2 \\ c &:= -3 & d &:= 2 \end{aligned}$$

Then:

$$R := M \cdot r \quad R = \begin{pmatrix} 6 \\ -2 \end{pmatrix}$$

In general there are two geometric interpretations of this transformation:



Using the left interpretation, suppose the  $(x,y)$  plane is covered by an elastic membrane which may be shrunk, stretched, or rotated so that:

- $r(x_r, y_r)$  becomes  $R(x_R, y_R)$  for any point on the plane

Using the right interpretation, suppose the points do not change position, but the reference system used to describe their location is shifted by a rigid rotation. In this case:

- $r(x_r, y_r)$  and  $R(x_R, y_R)$  are the same but the coordinate in the reference system for  $r$  and  $R$  have changed

Using the left geometric interpretation above, we may ask if there are any vectors  $r$  that remain unchanged in direction during transformation? From Multivariate Worksheet MTB 040, we note that in fact this is the case. These are termed the "principal" or "characteristic" directions. Using the right interpretation above, Eigenvectors can be visualized as having their foot at the center of the reference system and head parallel to the directions of reference system  $R$ .

Note, however, that we will allow the space to be shrunk or stretched in the Eigenvector directions. The amount of this stretching is called the "principal" or "characteristic" values calculated for each Eigenvector as an associated Eigenvalue.

In algebraic terms, vector Eigenvector  $R = \lambda r$  where Eigenvalue  $\lambda$  is some scalar constant.

$$R = \lambda \cdot r$$

$$\begin{pmatrix} x_R \\ y_R \end{pmatrix} = \lambda \cdot \begin{pmatrix} x_r \\ y_r \end{pmatrix} = \begin{pmatrix} \lambda \cdot x_r \\ \lambda \cdot y_r \end{pmatrix}$$

To calculate eigenvalues and eigenvectors, remember that:

$$R = M \cdot r = \lambda \cdot r$$

$$R = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} x_r \\ y_r \end{pmatrix} = \lambda \cdot \begin{pmatrix} x_r \\ y_r \end{pmatrix}$$

$$R = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} x_r \\ y_r \end{pmatrix} = \begin{pmatrix} \lambda \cdot x_r \\ \lambda \cdot y_r \end{pmatrix}$$

Converting matrix algebra to simultaneous equations:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} x_r \\ y_r \end{pmatrix} = \begin{pmatrix} \lambda \cdot x_r \\ \lambda \cdot y_r \end{pmatrix}$$

$$a \cdot x_r + b \cdot y_r = \lambda \cdot x_r \quad \text{or} \quad (a - \lambda) \cdot x_r + b \cdot y_r = 0$$

$$c \cdot x_r + d \cdot y_r = \lambda \cdot y_r \quad \text{or} \quad c \cdot x_r + (d - \lambda) \cdot y_r = 0$$

$$\text{Eigenvector equation} > \begin{pmatrix} a - \lambda & b \\ c & d - \lambda \end{pmatrix} \cdot \begin{pmatrix} x_r \\ y_r \end{pmatrix} = 0$$

For  $x_r$  &  $y_r$  both not equal to zero, the eigenvector equation is solved using the determinant:

$$\left| \begin{pmatrix} a - \lambda & b \\ c & d - \lambda \end{pmatrix} \right| = 0 \quad < \text{Characteristic equation of M}$$

$$(a - \lambda) \cdot (d - \lambda) - bc = 0$$

$$a \cdot d - a \cdot \lambda - d \cdot \lambda + \lambda^2 - b \cdot c = 0$$

$$(1) \cdot \lambda^2 - (a + d) \cdot \lambda + (a \cdot d - b \cdot c) = 0 \quad < \text{Quadratic equation of the general form:}$$

$$A \cdot \lambda^2 + B \cdot \lambda + C = 0$$

**General solution to the quadratic:**

$$\lambda = \frac{-B + \sqrt{B^2 - 4 \cdot A \cdot C}}{2 \cdot A} \quad \text{or} \quad \lambda = \frac{-B - \sqrt{B^2 - 4 \cdot A \cdot C}}{2 \cdot A}$$

**Thus:**  $(1) \cdot \lambda^2 - (a + d) \cdot \lambda + (a \cdot d - b \cdot c) = 0$

**has two general solutions:**

$$\lambda = \frac{(a + d) + \sqrt{(a + d)^2 - 4 \cdot (1) \cdot (a \cdot d - b \cdot c)}}{2 \cdot (1)} \quad \text{or} \quad \lambda = \frac{(a + d) - \sqrt{(a + d)^2 - 4 \cdot (1) \cdot (a \cdot d - b \cdot c)}}{2 \cdot (1)}$$

**To calculate eigenvectors, replace each solved eigenvalue back into the Eigenvector equation.**

**Note:** Equations will be redundant and will therefore specify the equation of a line.

As a result, it is customary give eigenvectors unit length.

**Things to remember about eigenvectors:**

- Eigenvectors found from M are not necessarily perpendicular.
- If M is symmetric [i.e.,  $M = M^T$ ] then eigenvectors are perpendicular.  
Only in this case is the right-hand picture above is a viable geometric interpretation.

**Example Calculation:**

**Solving for Eigenvalues:**

$$M = \begin{pmatrix} 5 & -2 \\ -3 & 2 \end{pmatrix} \quad \begin{array}{l} a := 5 \quad b := -2 \\ c := -3 \quad d := 2 \end{array}$$

$$\lambda_1 := \frac{(a + d) + \sqrt{(a + d)^2 - 4 \cdot (1) \cdot (a \cdot d - b \cdot c)}}{2 \cdot (1)} \quad \lambda_1 = 6.372$$

$$\lambda_2 := \frac{(a + d) - \sqrt{(a + d)^2 - 4 \cdot (1) \cdot (a \cdot d - b \cdot c)}}{2 \cdot (1)} \quad \lambda_2 = 0.628$$

**Solving for Eigenvectors:**

$$\begin{pmatrix} a - \lambda_1 & b \\ c & d - \lambda_2 \end{pmatrix} \cdot E = 0$$

^ Eigenvector equation

**Plugging Eigenvalues back into the Eigenvector equation results in expressions that give the x coordinate of each Eigenvector in terms of y or vice versa. This is the equation for a line and implies a constant ratio between y and x for each Eigenvector.**

**For Eigenvector 1:**

$$(a - \lambda_1) \cdot x + b \cdot y = 0$$

$$x = \frac{-b}{(a - \lambda_1)} \cdot y$$

$$\frac{-b}{(a - \lambda_1)} = -1.4574271$$

^ ratios of x in terms of y ^

**For Eigenvector 2:**

$$c \cdot x + (d - \lambda_2) = 0$$

$$x = \frac{d - \lambda_2}{c} \cdot y$$

$$\frac{-(d - \lambda_2)}{c} = 0.4574271$$

**Mathcad's built-in functions:**

$\text{evals} := \text{eigenvals}(M)$	$\text{evals} = \begin{pmatrix} 6.3722813 \\ 0.6277187 \end{pmatrix}$	< Vector of eigenvalues
$\text{evecs} := \text{eigenvecs}(M)$	$\text{evecs} = \begin{pmatrix} 0.8245648 & 0.4159736 \\ -0.5657675 & 0.9093767 \end{pmatrix}$	< Matrix of eigenvectors, each column is an eigenvector associated with one of the eigenvalues above. Generally, each column is in the same order as the elements in the eigenvalue vector.
$\text{eigenvec}(M, \text{evals}_1) = \begin{pmatrix} -0.825 \\ 0.566 \end{pmatrix}$	$\text{eigenvec}(M, \text{evals}_2) = \begin{pmatrix} 0.416 \\ 0.909 \end{pmatrix}$	< A way to check which eigenvector belongs with which eigenvalue!

Note that Eigenvector 1 and Eigenvector 2 have the same ratios of x versus y values as calculated above:

**For Eigenvector 1:**

$$-1.4574271 \cdot -0.5657675 = 0.8245649$$

^ratio   ^y   ^x

**For Eigenvector 2:**

$$0.4574271 \cdot 0.9093767 = 0.41597355$$

^ratio   ^y   ^x

**Prototype in R:**

**#EIGENVECTORS AND EIGENVALUES:**

**#MAKING MATRIX M:**

```
M=matrix(c(5,-2,-3,2),nrow=2,ncol=2,byrow=T)
```

```
> M
```

```
      [,1] [,2]
[1,]    5  -2
[2,]   -3    2
```

```
>
```

**#COMPUTING EIGENVALUES AND EIGENVECTORS OF MATRIX M:**

```
E=eigen(M)
```

```
> E
```

```
$values
```

```
[1] 6.3722813 0.6277187
```

```
$vectors
```

```
      [,1] [,2]
[1,] 0.8245648 0.4159736
[2,] -0.5657675 0.9093767
```

**Length of reported Eigenvectors:**

**For Eigenvector 1:**

$$E_1 := \text{evecs}^{(1)} \quad E_1 = \begin{pmatrix} 0.8245648 \\ -0.5657675 \end{pmatrix}$$

$$L_{E1} := \left| \sqrt{E_1^T \cdot E_1} \right| \quad L_{E1} = 1$$

**For Eigenvector 2:**

$$E_2 := \text{evecs}^{(2)} \quad E_2 = \begin{pmatrix} 0.4159736 \\ 0.9093767 \end{pmatrix}$$

$$L_{E2} := \left| \sqrt{E_2^T \cdot E_2} \right| \quad L_{E2} = 1$$

Because solution of the Eigenvector equation only results in equation of lines, Eigenvectors have determined direction (ratio of x versus y) but are arbitrary in length. It is customary to scale Eigenvectors to unit length. As you can see, both MathCad and R did this, resulting in the numerical values reported for the Eigenvectors in each case.

## Displaying Eigenvectors:

Here for visual effect we use less extreme matrix M from Multivariate Worksheet MTB 040:

$$M := \begin{pmatrix} 1 & 0.3 \\ 0.3 & 1 \end{pmatrix}$$

(function `make.grid()` included in R script)

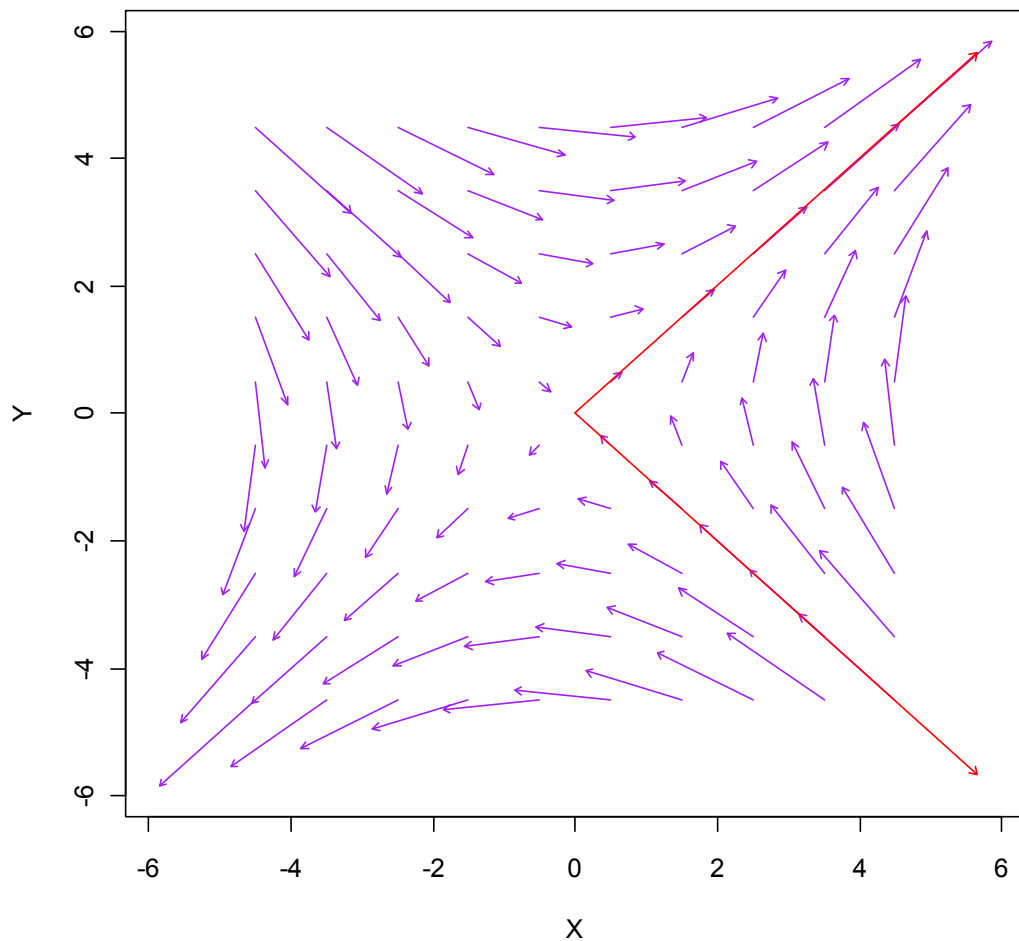
```
X=make.grid(10,10)
M=matrix(c(1.0,0.3,0.3,1),nrow=2,ncol=2,byrow=T)
E=eigen(M)
```

**#LINEAR TRANSFORMATION:**

```
Z=M%*%t(X)
Z=t(Z)
```

**#PLOT TRANSFORMATION VECTORS:**

```
plot(Z,type='n',xlab='X',ylab='Y')
arrows(X[,1],X[,2],Z[,1],Z[,2],col='purple',lwd=1,code=2,length=0.05)
arrows(0,0,8*E$eigenvectors[,1],8*E$eigenvectors[,2],col='red',code=2,length=0.05)
```



Eigenvectors in red (scaled to a length of 8 for display effect).